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Unification and higher-derivative gravity

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Abstract. We consider higher-derivative theories of gravitation in which βR^2 and $\alpha R_{\mu\nu}R^{\mu\nu}$ terms are present in the Lagrangian density, in addition to the usual scalar curvature R term. We show that using this Lagrangian to obtain the field equations in a fibre-bundle unification of electromagnetism and general relativity leads to unacceptable modifications to Maxwell's equations even for very small α. Since the Kaluza–Klein unification and related fibre-bundle unifications of general relativity with a non-Abelian gauge group are major successes of the geometrical ideas behind general relativity, these results suggest that adding an $R_{\mu\nu}R^{\mu\nu}$ term is not a profitable approach to future gravitational theories. The R^2 term, on the other hand, is not ruled out by these considerations.

1. Introduction

Higher-derivative theories of gravitation, in which R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms appear in the Lagrangian density in addition to the usual scalar curvature R, have been suggested by various authors (Utiyama and DeWitt 1962, Sakharov 1967, Deser 1976). Interest in these theories has grown considerably since Stelle (1977) showed that they are renormalisable. As he points out, however, these theories involve massive spin-two excitations with negative definite energy. In the quantum theory these states can be cast into a form with positive definite energy but with negative norm. The negative norm states, in turn, lead to unitarity problems (Pais and Uhlenbeck 1950). These unitarity problems must be resolved before a sensible physical interpretation can be given. There are also a variety of other undesirable features in these theories (Pechlaner and Sexl 1966, Buchdahl 1970, Ruzmaikina and Ruzmaikin 1970, Michel 1973, Havas 1977).

These theories, however, have other nice properties in addition to renormalisability. Macrae (1981) showed that these higher-derivative terms give a Euclidean functional integral convergent on the metric conformal factor. Tomboulis (1980) found that quantum gravity is asymptotically free if these terms are included. Other work includes an investigation of the effect of these terms on cosmology by Macrae and Riegert (1981).

In the present work we will investigate the effects of R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms on the unification of general relativity with the electromagnetic field. One of the major conceptual ingredients of general relativity is the idea of the geometrisation of physics. Over 50 years ago Kaluza (1921) and Klein (1921) showed that general relativity could be unified with electromagnetism in a five-dimensional geometry using the scalar curvature R of this manifold as the Lagrangian density. The idea of geometrisation lay fallow until recently when it has begun to bear fruit in the geometrical fibre-bundle approach to gauge theories. Kerner (1968) and Cho (1975) put the work of Kaluza and Klein on a firmer mathematical footing using fibre bundles and generalised it to an n-dimensional non-Abelian gauge group. The Lagrangian density is again the scalar curvature of the $(4+n)$-dimensional fibre bundle with space–time as the base space. Rather miraculously, this even works for supergravity, using four anticommuting generators in the gauge space (Ross 1979, 1981). The self-consistency of these unifications can be viewed as one of the major theoretical successes of general relativity and its scalar curvature Lagrangian density.

In the present work we will investigate what happens to a Kaluza–Klein type unification if R^2 and $R_{AB}R^{AB}$ are also included in the Lagrangian density of the fibre bundle. Specifically, we will consider the electromagnetic case with U(1) as the structure group and space–time as the base space of the fibre bundle. The variational principle for the field equations will be taken to be

$$\delta \int d^5x(-\gamma)^{1/2}(R/16\pi G+\alpha R_{AB}R^{AB}-\beta R^2)=0 \tag{1}$$

where γ is the trace of the metric for the five-dimensional fibre-bundle space. R and R_{AB} are the scalar curvature and the Riemann curvature tensor for this five-dimensional space. (Greek indices refer to space–time while A, B, C refer to the five-dimensional fibre-bundle space.) If $\alpha\equiv\beta\equiv 0$, we get the usual Kaluza–Klein theory.

We show below that the Maxwell equations resulting from (1) are modified in an unacceptable way *even for very small* α so that the unification of general relativity and electromagnetism is spoiled. This result strongly suggests that $R_{\mu\nu}R^{\mu\nu}$ terms should *not* be included and that such terms are not a promising avenue for future theories of gravitation.

We will define our fibre bundle and work out the new unified field equations in § 2 below. In § 3 we look at the Maxwell sector of these equations in detail and in § 4 we draw our conclusions.

2. Fibre-bundle structure and field equations

We consider a fibre bundle with U(1) as the structure group and space–time with metric $g_{\mu\nu}$ (signature $-1, +1, +1, +1$) as the base space. We can choose a coordinate basis $\xi_\mu=\partial_\mu$ for the base space with commutation relations

$$[\xi_\mu,\xi_\nu]=0. \tag{2}$$

For a basis for the gauge group we can choose a left-invariant vector field ξ_1 which can also be viewed as a basis of the Lie algebra. The commutation relations

$$[\xi_1,\xi_1]=0 \tag{3}$$

are trivial for U(1). A subscript or superscript 1 always refers to the gauge space or vertical subspace of the fibre bundle. In the five-dimensional fibre bundle, we will work in the horizontal lift basis (Cho 1975) for convenience where the commutation relations become

$$[\xi_1^*,\xi_1^*]=0, \qquad [\hat{\xi}_\mu,\hat{\xi}_\nu]=-F^1_{\mu\nu}\xi_1^*, \qquad [\hat{\xi}_\mu,\xi_1^*]=0. \tag{4}$$

$F^1_{\mu\nu}$ can be written as the usual electromagnetic field tensor $F_{\mu\nu}$ to within a units transformation. We will often delete the 1 superscript in the following. In this basis, the fibre-bundle metric γ_{AB} is particularly simple and is given by

$$\gamma_{AB}=\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{11}\end{pmatrix}. \tag{5}$$

Without loss of generality we can take the U(1) metric g_{11} to be a constant and in particular 1. The usual choice in terms of structure constants, of course, is not possible. We pay the price for using this non-coordinate basis in extra terms in the Christoffel symbols,

$$\Gamma^{E}_{AB}=\tfrac{1}{2}(\gamma_{AD,B}+\gamma_{BD,A}-\gamma_{AB,D})\gamma^{DE}-\tfrac{1}{2}H^{E}{}_{BA}-\tfrac{1}{2}\gamma_{CB}H^{C}{}_{DA}\gamma^{DE}-\tfrac{1}{2}\gamma_{AC}H^{C}{}_{DB}\gamma^{DE} \tag{6}$$

where $H^{1}_{\mu\nu}\equiv F^{1}_{\mu\nu}$ and all other H^{C}_{AB} vanish. Working this out gives

$$\Gamma^{\nu}_{\mu\gamma}=\tfrac{1}{2}g^{\delta\nu}[g_{\mu\delta,\gamma}+g_{\gamma\delta,\mu}-g_{\mu\gamma,\delta}], \qquad \Gamma^{1}_{\mu\nu}=\tfrac{1}{2}F^{1}_{\mu\nu}, \qquad \Gamma^{\nu}_{1\mu}=\Gamma^{\nu}_{\mu 1}=\tfrac{1}{2}F_{\mu}{}^{\nu} \tag{7}$$

with all other Γ^{A}_{BD} vanishing. Ordinary derivatives with respect to the 1 coordinate always vanish. For completeness, we also have

$$R_{FG}=\Gamma^{A}_{FG,A}-\Gamma^{A}_{FA,G}+\Gamma^{C}_{FG}\Gamma^{A}_{CA}-\Gamma^{C}_{FA}\Gamma^{A}_{CG}-\Gamma^{A}_{FE}H^{E}_{GA} \tag{8}$$

which works out in the various sectors as

$$R_{\mu\nu}=R^{(\mathrm{E})}_{\mu\nu}-\tfrac{1}{2}F_{\mu\alpha}F_{\nu}{}^{\alpha}, \qquad R=R^{(\mathrm{E})}-\tfrac{1}{4}F_{\mu\alpha}F^{\mu\alpha},$$
$$R_{\mu 1}=R_{1\mu}=-\tfrac{1}{2}F^{\alpha}{}_{\mu;\alpha}, \qquad R_{11}=\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{9}$$

where (E) refers to the usual Einstein expression and a semicolon denotes the usual covariant derivative.

It should be emphasised that even though we work in the horizontal lift basis, any other basis would give equivalent final results. Kerner (1968), for example, works in the local direct product basis, which is a coordinate basis with a more complicated form for γ_{AB}.

To obtain field equations, we will assume the Lagrangian given in (1). α and β are constants. Note that the only other possible quadratic term is $R_{ABCD}R^{ABCD}$. This need not be included (Stelle 1977) since it can always be written in terms of $R_{AB}R^{AB}$ and R^2. We also note that no source terms are included in (1) in the spirit of unification. These of course could be inserted. Thus our Maxwell equations will come out sourceless, and the source of the gravitational field will only be the energy momentum tensor of the electromagnetic field.

We now want to carry out the variation indicated in (1) to get field equations. This is a rather lengthy calculation which yields the final result

$$\begin{aligned}\int \mathrm{d}x^5(-\gamma)^{1/2}\bigg\{&\frac{1}{16\pi G}(-R_{AB}+\tfrac{1}{2}\gamma_{AB}R)\\ &-\beta(-2RR_{AB}+\tfrac{1}{2}R^2\gamma_{AB}-2\gamma_{AB}R^{|C}{}_{\|C}+R_{A\|B}+R_{|B\|A})+\alpha(-2R_{AC}R_B{}^{C}\\ &+\tfrac{1}{2}R_{CD}R^{CD}\gamma_{AB}-\gamma_{AB}R^{CD}{}_{\|D\|C}-R_{AB\|C\|D}\gamma^{CD}\\ &+R^{C}{}_{A\|B\|C}+R^{C}{}_{B\|A\|C})\bigg\}\,\delta\gamma^{AB}=0,\end{aligned} \tag{10}$$

where a single vertical bar denotes ordinary differentiation and a double vertical bar denotes covariant differentiation in the five-dimensional fibre-bundle space. Note that the expression in the braces is symmetric under $A\leftrightarrow B$. We now have to be a bit careful. The variation δg^{AB} is not completely arbitrary but must preserve the original form of the metric (5) with $g_{11} = \text{constant}$. This is most easily accomplished by setting $\delta g^{11} = 0$. An equivalent and somewhat more elegant procedure is to formally integrate out the gauge degree of freedom in (1) as Cho (1975) does. The same thing, of course, is done in the usual Kaluza–Klein theory. In that case $\alpha \equiv \beta \equiv 0$ in (10). Using (9) it is readily apparent that the μ, ν sector in (10) gives the Einstein equations correctly coupled to the electromagnetic energy momentum tensor. (Letting $F^1_{\mu\nu} \to (16\pi G/c^4)^{1/2} F_{\mu\nu}$ gives conventional units for $F_{\mu\nu}$.) The μ, 1 sector gives $F^{\mu\nu}{}_{;\nu} = 0$ and the 1, 1 sector does not contribute since $\delta g^{11} = 0$. If g^{11} were allowed to vary, the unphysical equation $F_{\mu\nu}F^{\mu\nu} = 0$ would result.

In the present case for non-zero α and β, using $\delta g^{11} = 0$ in (10) with the other metric variations arbitrary gives the μ, ν sector field equations

$$\begin{aligned}(16\pi G)^{-1}(-R_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}R) - \beta(&-2RR_{\mu\nu} + \tfrac{1}{2}R^2 g_{\mu\nu} \\ &- 2g_{\mu\nu}R^{|C}{}_{\|C} + R_{|\mu\|\nu} + R_{|\nu\|\mu}) + \alpha(-2R_{\mu C}R_{\nu}{}^{C} + \tfrac{1}{2}R_{CD}R^{CD}g_{\mu\nu} \\ &- g_{\mu\nu}R^{CD}{}_{\|D\|C} - R_{\nu\mu\|C\|D}\gamma^{CD} + R^{C}{}_{\mu\|\nu\|C} + R^{C}{}_{\nu\|\mu\|C}) = 0 \end{aligned} \qquad (11)$$

and the μ, 1 sector field equations

$$\begin{aligned}(-R_{\mu 1})(16\pi G)^{-1} - \beta(&-2RR_{\mu 1} + R_{|\mu\|1} + R_{|1\|\mu}) \\ &+ \alpha(-2R_{\mu C}R_{1}{}^{C} - R_{\mu 1\|C\|D}\gamma^{CD} + R^{C}{}_{\mu\|1\|C} + R^{C}{}_{1\|\mu\|C}) = 0. \end{aligned} \qquad (12)$$

R and $R_{\mu\nu}$ in these equations are given by (9). The remaining equation for the electromagnetic field arises from the fact that $\hat{\xi}_\mu$ and ξ_1^* must satisfy the Jacobi identity. This gives

$$F_{\nu\delta|\mu} + F_{\delta\mu|\nu} + F_{\mu\nu|\delta} = 0 \qquad (13)$$

independent of the Lagrangian assumed.

3. Modified Maxwell equations

We now want to work out the field equations (11) and (12) in terms of $R^{(\mathrm{E})}_{\mu\nu}$, $R^{(\mathrm{E})}$, $g_{\mu\nu}$ and $F_{\mu\nu}$. We are interested in seeing whether these field equations are sensible or whether they are at variance with experiment. Equation (11) is not very useful for this purpose. It reduces to the usual Einstein equation, correctly coupled to the electromagnetic energy momentum tensor in the low-energy limit. Since even this low-energy equation with electromagnetic sources has not been tested by experiment, the experimental testing of correction terms is out of the question. For this reason we will concentrate on the Maxwell sector and equation (12). If $\alpha \equiv \beta \equiv 0$ this just gives the sourceless Maxwell equation $F^{\mu\nu}{}_{;\nu} = 0$. We are interested in the correction terms arising for non-zero α and β.

Using (9) and (7) along with the definition of a covariant derivative lets us write out (12) after a long, tedious calculation as

$$\mathscr{F}^{\mu\beta}{}_{;\beta} = 0, \qquad (14)$$

where the semicolon denotes the usual covariant derivative and

$$\begin{aligned}\mathscr{F}^{\mu\beta} \equiv -F^{\mu\beta}(32\pi G)^{-1} &+ \beta(R^{(\mathrm{E})} - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta})F^{\mu\beta} \\ &+ \alpha(\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}F^{\mu\beta} + R^{(\mathrm{E})\mu}{}_{\pi}F^{\beta\pi} - R^{(\mathrm{E})\beta}{}_{\pi}F^{\mu\pi} - \tfrac{1}{2}F^{\mu\pi}{}_{;\pi}{}^{;\beta} + \tfrac{1}{2}F^{\beta\pi}{}_{;\pi}{}^{;\mu}). \end{aligned} \qquad (15)$$

$\mathscr{F}^{\mu\beta}$ is completely antisymmetric. If a source current were added on the right-hand side of (14), this current would be conserved identically from this antisymmetry. We would like to see if (15) has observational consequences. First, let us replace $F_{\delta\mu}$ by $(16\pi G/c^4)^{1/2}F_{\delta\mu}$ in order to put the electromagnetic field tensor in conventional units and define

$$l_\alpha^2 = 32\pi G\alpha, \qquad l_\beta^2 = 32\pi G\beta \tag{16}$$

where l_α and l_β have dimensions of length. The Newtonian limit of the static field requires that $3\beta \geq \alpha \geq 0$ and that α and β be small or unacceptable modifications of Newtonian gravity would result (Stelle 1977). Thus l_α and l_β defined in (16) are real lengths. We can also define $\acute{\mathscr{F}}^{\mu\beta} \equiv -(64\pi G c^4)^{1/2}\mathscr{F}^{\mu\beta}$ to get

$$\begin{aligned}\acute{\mathscr{F}}^{\mu\nu} \equiv F^{\mu\nu} &- l_\beta^2(R^{(\mathrm{E})} - F_{\delta\rho}F^{\delta\rho}\, 4\pi G/c^4)F^{\mu\nu} \\ &- l_\alpha^2[F_{\delta\rho}F^{\delta\rho}(4\pi G/c^4)F^{\mu\nu} + R^{\mu}{}_{\delta}F^{\nu\delta} - R^{\nu}{}_{\delta}F^{\mu\delta} - \tfrac{1}{2}F^{\mu\delta}{}_{;\delta}{}^{;\nu} + \tfrac{1}{2}F^{\nu\delta}{}_{;\delta}{}^{;\mu}].\end{aligned} \tag{17}$$

Let us look at this in the limit of nearly flat space and laboratory scale electromagnetic fields. If we also assume that l_α and l_β are small and comparable in magnitude, we find that the dominant correction terms are the last two terms in (17). Then our field equation (14) can be written

$$F^{\mu\beta|}{}_{\beta} = -\tfrac{1}{2}l_\alpha^2(F^{\mu\delta}{}_{|\delta}{}^{|\beta}{}_{|\beta} - F^{\beta\delta}{}_{|\delta}{}^{|\mu}{}_{|\beta}), \tag{18}$$

where we now have ordinary derivatives. The last term in (18) vanishes since $F^{\beta\delta}$ is antisymmetric. Equation (18) can then be written in the form

$$Q^\mu + \tfrac{1}{2}l_\alpha^2 \Box Q^\mu = 0, \tag{19}$$

where $Q^\mu \equiv F^{\mu\beta}{}_{|\beta}$. Note that we have the constraint $Q^\mu{}_{|\mu} \equiv 0$. The solution to this Klein–Gordon type equation is

$$Q^\mu \equiv F^{\mu\beta}{}_{|\beta} = A^\mu \sin k^\delta x_\delta + B^\mu \cos k^\delta x_\delta \tag{20}$$

where $k^\delta k_\delta = 2/l_\alpha^2$ and $A^\delta k_\delta = B^\delta K_\delta = 0$ from the constraint. We now note that even though we have *no sources* in the theory, nonetheless (20) looks like the usual Maxwell equation with a four-current

$$J^\mu \equiv A^\mu \sin k^\delta x_\delta + B^\mu \cos k^\delta x_\delta. \tag{21}$$

This strange travelling wave current can have (a) large amplitude components, (b) laboratory scale wavelengths and (c) laboratory scale frequencies depending on the choice of parameters *even if l_α gets very, very small.* This current is obviously ruled out by experiment. The fact that Maxwell's equations are modified in an unacceptable way even for very small values of l_α or of α in the original Lagrangian (1) is a reflection of the fact that the limit $l_\alpha \to 0$ is a highly singular point for the differential equation (19). The l_β^2 terms in (17) do not contribute to the lowest-order modifications of the Maxwell equations looked at above. Thus βR^2 terms can enter the Lagrangian (1) without dramatic effects on the Maxwell equations. However, in many considerations such as asymptotic freedom (Tomboulis 1980), for example, α and β come in together in a particular combination so that eliminating α terms and leaving β terms in (1) would spoil their argument.

4. Conclusion

We found above that if the Lagrangian (1) is used in a Kaluza–Klein type unification of gravity and electromagnetism, then the $\alpha R_{AB}R^{AB}$ term in the Lagrangian modifies the resulting Maxwell equations in an unacceptable way even for very, very small α. Since the Kaluza–Klein unification and related unifications using non-Abelian gauge groups in fibre bundles are major successes of the idea of the geometrisation of physics embodied in general relativity, this result suggests that modifying general relativity by adding such a term to the Lagrangian is not a profitable approach to future gravitational theories. The βR^2 term in (1), on the other hand, is not ruled out by these considerations.

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